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NON-LINEAR DYNAMICS OF ELASTIC CURVILINEAR SYSTEMS. A NEW COORDINATELESS APPROACH†

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Using the theory of Lie groups, a mechanical model is developed and the equations of motion are written in coordinateless form for an arbitrary curvilinear model of an elastic system with dimension no higher than three, starting from the unique hypothesis of a continuous distribution of rigid bodies, called sections. The proposed method is illustrated using examples of the automatic derivation of the corresponding scalar equations, by means of which one can describe all possible models of such systems as beams, cables, strings, etc.

Many mechanical systems contain curvilinear components, and different mechanical models (beams, cables or strings) are used to describe them. The derivation of the equations of motion of such components, both for low strains and stresses and in the general case, is an important step when investigating such systems. The main purpose of the present paper is to use calculations based on Lie groups to obtain the equations of motion in a general and in a coordinateless form. As was shown in other cases (see, for example, [1] when describing the motion of robots), such a coordinateless approach has a number of obvious advantages. As an example of these advantages we can mention the effective solution of problems related to the six-dimensional form of the problems considered, with a generality of the calculations irrespective of whether the strains and stresses are low or not, using limited expansions (for example, linearization) or an automatic derivation of the scalar equations of motion.

The discussions rest on a single mechanical hypothesis, in which the curvilinear system is regarded as a continuous distribution of rigid bodies, called sections of an object. A comparison with the mechanics of multibody systems, in particular, with the mechanics of chains, clearly demonstrates an analogy between the approach employed in it and the proposed approach. The difference is solely that, in the case of the mechanics of multibody systems, the kinematic, kinetic and dynamic quantities are functions of a single discrete variable, whereas in the case considered they depend on a continuous variable. The model proposed is the most general coordinateless model of a one-dimensional microstructure.

In Section 1 we briefly describe the mathematical apparatus of Lie groups. In Section 2 we describe the model of the system and, within framework of this system, we describe the kinematic, kinetic and dynamic properties of the system; here we derive the equations of motion in coordinateless form. Section 3 is devoted to the relations between the equations obtained and the classical models, such as a string, a beam or a cable. In Section 4 we consider problems of the automatic derivation of the equivalent scalar equations by parametrizing the variables and, mainly, rotations; the complexity of these scalar non-linear equations proves a posteriori the effectiveness of the coordinateness approach.

1. MATHEMATICAL APPARATUS

Suppose ξ is a three-dimensional affine space with the usual properties of Euclidean geometry and E is an associated vector space. For each affine mapping $A\colon \mathscr{E} \to \mathscr{E}$ there is usually a linear part $\mathbf{A}$ corresponding to it, so that

$$A(m) = A(p) + \mathbf{A}(\overrightarrow{pm}), \quad \forall m, p \in \mathscr{E}$$

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We will denote by $\mathbb{D}$ the group of affine mappings A such that $\mathbf{A}$ is an element of a special orthogonal group SO(3). Suppose $\mathfrak{D}$ is a six-dimensional vector space of skewsymmetric vector fields $X: \mathscr{E} \to E$ such that in E there is a vector $\boldsymbol{\omega}_X$ with the following well-known property

$$X(a) = X(b) + \boldsymbol{\omega}_X \wedge \overrightarrow{ba}, \quad \forall a, b \in \mathscr{E}$$

In other words, the linear part $\mathbf{X}$ of the field X is the following linear operator in E

$$u \mapsto \mathbf{X}(u) = \boldsymbol{\omega}_X \wedge u$$

and $\mathfrak{D}$ is identified with the set of screws.

The Lie bracket is defined in $\mathfrak{D}$ by the relation

$$[X, Y](a) = \boldsymbol{\omega}_X \wedge Y(a) - \boldsymbol{\omega}_Y \wedge X(a), \quad a \in \mathscr{E}$$

Hence, $\mathfrak{D}$ is a Lie algebra, isomorphic with classic Lie algebra of the space $\mathbb{D}$ and identified with it.

The exponential mapping exp: $\mathfrak{D} \to \mathbb{D}$ enables us to express a finite variable using its infinitesimal operator, and the adjoint mapping Ad: $\mathbb{D} \to \mathscr{L}(\mathfrak{D})$ describes the action of displacements on the elements of the space $\mathfrak{D}$. By virtue of duality, each action of the rigid body will be described by element of the space $\mathfrak{D}$, while the operator, which allows of this identification, represents the classical internal product $[\cdot|\cdot]$ (the Killing form of the theory of Lie groups), defined as

$$[X|Y] = (\boldsymbol{\omega}_X|Y(p)) + (\boldsymbol{\omega}_Y|X(p))$$

with a right-hand side which is independent of the point p in the space ξ. For refinement of the details touching on the operations defined in the space $\mathbb{D}$ and $\mathfrak{D}$, see [2].

2. THE MODEL OF THE SYSTEM

Kinematics. The system is described as follows. The initial configuration is similar to the distribution

$$\sigma \mapsto r(\sigma) = (A(\sigma); \mathbf{i}_1(\sigma), \mathbf{j}_1(\sigma), \mathbf{k}_1(\sigma))$$

of affine frames of reference, where σ is the curvilinear abscissa along the curve $\sigma \mapsto A(\sigma)$, where $A(\sigma)$, for example, is the centre of inertia of the section for the given abscissa σ, while $\mathbf{i}_1(\sigma)$, $\mathbf{j}_1(\sigma)$, $\mathbf{k}_1(\sigma)$ is the basis connected with the rigid section for this abscissa σ. We can take as this basis, for example, the Frenet trihedron $\sigma \mapsto A(\sigma)$. Henceforth each rigid section and the system of coordinates associated with it will be identified.

An unknown displacement $D(\sigma, t)$ acts on each section $r(\sigma)$ at each instant of time t, such that

$$r(\sigma) \to r_a(\sigma, t) = D(\sigma, t) \bullet r(\sigma), \quad r_a(\sigma, t) = (a(\sigma, t); \mathbf{i}_2(\sigma, t), \mathbf{j}_2(\sigma, t), \mathbf{k}_2(\sigma, t))$$

Note that no assumption is made regarding the perpendicularity of the section of the curve $(\sigma, t) \mapsto a(\sigma, t)$. The dark dot denotes the natural action of $\mathbb{D}$ on the set of affine bases (in mathematical terms there is a stratified beam structure).

The kinematics of the system is specified by the velocity field

$$v^c: [0, l] \times \mathbb{R}^+ \to \mathfrak{D}: (\sigma, t) \mapsto v^c(\sigma, t) = \mathbf{D}(\sigma, t)^{-1} \circ \frac{\partial D(\sigma, t)}{\partial t}$$

and the acceleration field

$$\dot{v}^c: [0, l] \times \mathbb{R}^+ \to \mathfrak{D}: (\sigma, t) \mapsto \dot{v}^c(\sigma, t) = \frac{\partial v^c(\sigma, t)}{\partial t}$$

and the strain field

$$e^c: [0, l] \times \mathbb{R}^+ \to \mathfrak{D}: (\sigma, t) \mapsto e^c(\sigma, t) = \mathbf{D}(\sigma, t)^{-1} \circ \frac{\partial D(\sigma, t)}{\partial \sigma}$$

The small circle denotes composition of the mappings.

Note that the field $\sigma \mapsto e^c(\sigma, t)$ remains unchanged for superposition with motion as a solid whole. We will assume that the motion is specified by the relations $D_1(t) \circ D(\sigma, t)$. Then

$$e_1^c(\sigma, t) = (\mathbf{D}_1(t) \circ \mathbf{D}(\sigma, t))^{-1} \circ \frac{\partial(D_1(t) \circ D(\sigma, t))}{\partial \sigma} =$$

$$= \mathbf{D}(\sigma, t)^{-1} \circ \mathbf{D}_1(t)^{-1} \circ \mathbf{D}_1(t) \circ \frac{\partial D(\sigma, t)}{\partial \sigma} = \mathbf{D}(\sigma, t)^{-1} \circ \frac{\partial D(\sigma, t)}{\partial \sigma} = e^c(\sigma, t)$$

Kinematics and dynamics. According to the chosen model we will assume that at each instant of time t and in each section $r_a(\sigma, t)$ we have: (1) the distribution $(\sigma, t) \mapsto \mathcal{T}(\sigma, t)$ of the moment fields, describing the external action, (2) the distribution $(\sigma, t) \mapsto \Theta(\sigma, t)$ of the moment fields describing the internal actions (action from the "right" with respect to the section σ of the part of the system on the "left"), (3) two concentrated forces at the ends $\mathcal{T}_0(t)$, $\mathcal{T}_l(t)$ (in some cases we can consider a family of concentrated forces $\mathcal{T}_k(t)$, applied in the section σ_k $(k = 1, \dots, n)$, which specifies discontinuities of the quantity $\sigma \mapsto \Theta(\sigma, t)$ and defines the behaviour on each element $([\sigma_k, \sigma_{k+1}])$, (4) the distribution $\sigma \mapsto \rho_0 = \rho_0(\sigma)$ of the mass density in the initial configuration, and (5) the distribution $\sigma \mapsto H_{r_a}(\sigma, t)$ of the operators $\mathfrak{D}$, describing the inertial actions (if necessary the distribution $\sigma \mapsto H_r(\sigma)$ of the linear operators of inertia in the initial configuration is also used), and we have the following correspondence between H_r and H_{r_a}

$$\mathrm{Ad}(D(\sigma, t)) \circ H_r(\sigma) \circ \mathrm{Ad}^{-1}(D(\sigma, t)) = H_{r_a}(\sigma, t)$$

General equations

Proposition 1. The equations of the system in the Lagrange description have the form (the arguments of σ and t are omitted)

$$\mathcal{T}^c = \rho_0 H_r(\dot{v}^c) + [v^c, \rho_0 H_r(v^c)] - [e^c, \Theta^c] - \frac{\partial \Theta^c}{\partial \sigma}$$
$$\mathcal{T}_0^c = \Theta^c(0), \quad \mathcal{T}_l^c = -\Theta^c(l) \tag{2.1}$$

where, for each object $U = U(\sigma, t)$ of the system the quantity $U^c = \mathrm{Ad}(D(\sigma, t)^{-1})U$ is the Lagrange expression for the function U, i.e. the function considered with respect to the initial configuration.

Proof. Consider the part of the system situated between σ and $\sigma + d\sigma$. It is in equilibrium in the current configuration if

$$\mathcal{T}(\sigma, t) = \rho_0 \frac{\partial H_{r_a}(\sigma, t)(\mathrm{Ad}(D(\sigma, t))(v^c))}{\partial t} - \frac{\partial \Theta}{\partial \sigma}$$

$$\mathrm{Ad}(D(\sigma, t))\mathcal{T}^c(\sigma, t) = \rho_0 \frac{\partial(\mathrm{Ad}(D(\sigma, t))H_r(\sigma)(v^c))}{\partial t} - \frac{\partial(\mathrm{Ad}(D(\sigma, t))\Theta^c)}{\partial \sigma} =$$

$$= \rho_0 \mathrm{Ad}(D(\sigma, t))(H_r(\dot{v}^c) + [v^c, H_r(v^c)]) - \mathrm{Ad}(D(\sigma, t))\left([e^c, \Theta^c] + \frac{\partial \Theta^c}{\partial \sigma}\right)$$

Using the biektive mapping $\mathrm{Ad}(D(\sigma, t))^{-1}$ we arrive at relation (2.1).

Remark. This coordinateless form of the equations enables us to identify the elements exactly and, in particular, their non-linear elastic properties. The appear in the terms $[v^c, \rho H_r(v^c)]$ and $[e^c, \Theta^c]$. Moreover, this formulation is of interest mainly in relation to the fact that differential calculations have already been carried out so that scalar equations are obtained.

We will now show how certain classical models can be included in the model considered.

3. SOME CLASSICAL MODELS

The general classical approach consists of choosing a Frenet trihedron $(\mathbf{a}(\sigma), \mathbf{t}(\sigma), \mathbf{n}(\sigma), \mathbf{b}(\sigma))$, which accompanies the curve $\sigma \mapsto \mathbf{a}(\sigma)$. Note that this choice, in general, connects three of the six degrees

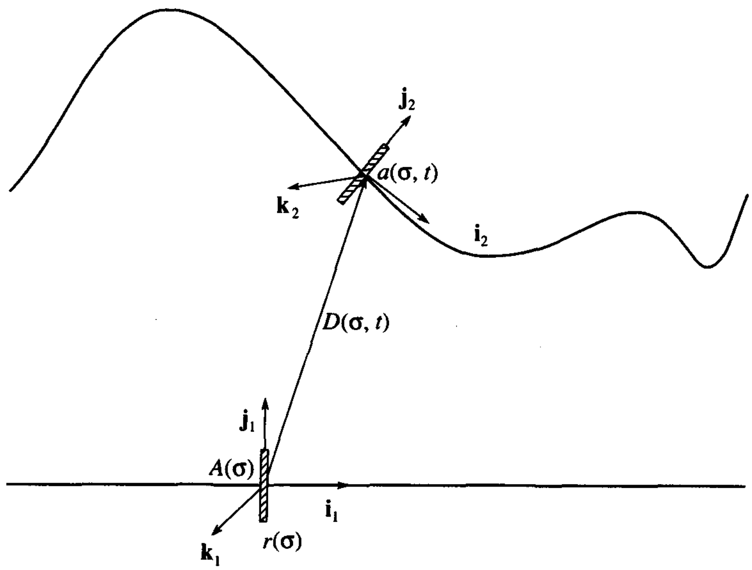

Fig. 1

of freedom for each of the sections. We will use the following classical notation, omitting the variable t from the arguments. We recall that we have the relations

$$\frac{d\mathbf{a}}{d\sigma} = \alpha(\sigma)\mathbf{t}(\sigma),\ \frac{d\mathbf{t}}{d\sigma} = \frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{n}(\sigma),\ \frac{d\mathbf{n}}{d\sigma} = -\frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{t}(\sigma) + \frac{\alpha(\sigma)}{\tau(\sigma)}\mathbf{b}(\sigma),\ \frac{d\mathbf{b}}{d\sigma} = -\frac{\alpha(\sigma)}{\tau(\sigma)}\mathbf{n}(\sigma)$$

where ν is the radius of curvature and τ is the radius of torsion of the curve.

With each system of coordinates $\mathcal{R} = (A; \mathbf{i}, \mathbf{j}, \mathbf{k})$ of the affine space ξ we associate a basis $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k}, \xi, \eta, \zeta)$ in the space $\mathcal{D}$ according to the following rule: for every $m \in \mathcal{E}$ we put

$$\begin{aligned} &\mathbf{i}(m) = \mathbf{i}, \quad \mathbf{j}(m) = \mathbf{j}, \quad \mathbf{k}(m) = \mathbf{k} \\ &\xi(m) = \mathbf{i} \wedge Am, \quad \eta(m) = \mathbf{j} \wedge Am, \quad \zeta(m) = \mathbf{k} \wedge Am \end{aligned} \tag{3.1}$$

We will use the basis $\mathcal{B}_1$ related to the system of coordinates $(A(\sigma); \mathbf{i}_1(\sigma), \mathbf{j}_1(\sigma), \mathbf{k}_1(\sigma))$, to describe the vectors of the Lie algebra $\mathcal{D}$ (see Fig. 1). We obtain the following proposition.

Proposition 2. If

$$r_a(\sigma, t) = (\mathbf{a}(\sigma, t); \mathbf{i}_2(\sigma, t), \mathbf{k}_2(\sigma, t)) = (\mathbf{a}(\sigma, t); \mathbf{t}(\sigma, t), \mathbf{n}(\sigma, t), \mathbf{b}(\sigma, t))$$

then

$$e^c = \text{col}[\alpha(\sigma) - 1, 0, 0, \alpha(\sigma)/\tau(\sigma), 0, \alpha(\sigma)/\nu(\sigma)]$$

Proof. We first note that

$$\frac{d\mathbf{a}}{d\sigma} = \alpha(\sigma)\mathbf{t}(\sigma) = \frac{\partial(D(\sigma, t)A(\sigma))}{\partial\sigma} = \frac{\partial D(\sigma, t)}{\partial\sigma}A(\sigma) + \mathbf{D}(\sigma, t)\mathbf{i}_1(\sigma)$$

Multiplying this equality by $\mathbf{D}(\sigma, t)^{-1}$, we obtain

$$\mathbf{D}(\sigma, t)^{-1}\frac{d\mathbf{a}}{d\sigma} = \mathbf{D}(\sigma, t)^{-1}\frac{\partial D(\sigma, t)}{\partial\sigma}A(\sigma) + \mathbf{D}(\sigma, t)^{-1}\mathbf{D}(\sigma, t)\mathbf{i}_1(\sigma) =$$

$$= \alpha(\sigma)\mathbf{D}(\sigma, t)^{-1}\mathbf{t}(\sigma) = e^c(A(\sigma)) + \mathbf{i}_1(\sigma) = \alpha(\sigma)\mathbf{i}_1(\sigma)$$

Moreover,

$$\frac{d\mathbf{t}(\sigma)}{d\sigma} = \frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{n}(\sigma) = \frac{\partial \mathbf{D}(\sigma, \mathbf{t})}{\partial\sigma}\mathbf{i}_1$$

since $\mathbf{i}_1$ is independent of σ. Multiplying the last equality by $\mathbf{D}(\sigma, t)^{-1}$, we obtain

$$\mathbf{D}(\sigma, t)^{-1}\frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{n}(\sigma) = \frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{D}(\sigma, t)^{-1}\mathbf{n}(\sigma) = \frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{D}(\sigma, t)^{-1}\mathbf{D}(\sigma, t)\mathbf{j}_1 = \frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{j}_1 = \boldsymbol{\omega}_{e^c} \wedge \mathbf{i}_1$$

In the same way we obtain

$$-\frac{\alpha(\sigma)}{\nu(\sigma)}\mathbf{i}_1 + \frac{\alpha(\sigma)}{\tau(\sigma)}\mathbf{k}_1 = \boldsymbol{\omega}_{e^c} \wedge \mathbf{j}_1, \quad -\frac{\alpha(\sigma)}{\tau(\sigma)}\mathbf{j}_1(\sigma) = \boldsymbol{\omega}_{e^c} \wedge \mathbf{k}_1$$

In turns out that the displacement field is bounded because three components of the vector e^c are equal to zero.

By duality, the three components of the vector Θ^c cannot be obtained using the defining relations. In fact, if we write the equations using the Virtual Power Theorem, these components of the vector Θ^c appear as Lagrange multipliers. Here, if the expression for Θ^c in the basis $\mathcal{B}_1$ has the form

$$\theta^c = \mathrm{col}[C, M_1, M_2, N, T_1, T_2]$$

then, by virtue of the chosen model the quantities T_1, T_2 and M_1 are none other than the previously considered components of the vector Θ^c. It should be added that this choice of the mobility of each of the section $r(\sigma)$ corresponds to the hypothesis that each of the sections remains during the motion perpendicular to the curve $\sigma \mapsto a(\sigma, t)$, which is the classical hypothesis of beam theory. Moreover, in the string theory, each section is assumed to be a point section. This imposes restrictions in kinematics, kinetics and dynamics. We will now discuss these propositions.

The classical theory of beams. We will assume that the motions are plane: motion occurs in the plane $(\mathbf{i}_1, \mathbf{j}_1)$ and the load is also situated in this plane. We will also assume that the previously assumed hypothesis that the sections are perpendicular holds.

The equation of statics has the form

$$\mathcal{T}^c + [e^c, \Theta^c] + \frac{\partial\Theta^c}{\partial\sigma} = 0$$

In the basis $\mathcal{B}_1$ we obtain

$$\left\|\begin{matrix} 0 \\ 0 \\ \Gamma \\ F \\ P \\ 0 \end{matrix}\right\| + \left\|\begin{matrix} \alpha(\sigma)-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha(\sigma)/\nu(\sigma) \end{matrix}\right\|, \left\|\begin{matrix} 0 \\ 0 \\ M \\ N \\ T \\ 0 \end{matrix}\right\| + \frac{\partial}{\partial\sigma}\left\|\begin{matrix} 0 \\ 0 \\ M \\ N \\ T \\ 0 \end{matrix}\right\| = 0 \tag{3.2}$$

We now consider the fact that the vectors from $\mathcal{B}_1$ are functions of σ so that $A = A(\sigma)$. It can be proved that if

$$u = x\mathbf{i}_1 + y\mathbf{j}_1 + z\mathbf{k}_1 + \lambda_1\xi_1 + \lambda_2\eta_1 + \lambda_3\zeta_1$$

then

$$\frac{du}{d\sigma} = x'\mathbf{i}_1 + (y' - \lambda_3)\mathbf{j}_1 + (z' + \lambda_2)\mathbf{k}_1 + \lambda_1'\xi_1 + \lambda_2'\eta_1 + \lambda_3'\zeta_1$$

Equations (3.2) take the form

$$\Gamma + \frac{dM}{d\sigma} + \alpha T = 0, \quad F + \frac{dN}{d\sigma} - \frac{\alpha}{\nu}T = 0, \quad P + \frac{dT}{d\sigma} + \frac{\alpha}{\nu}N = 0$$

where all the quantities are functions of σ.

If the beam is rectilinear, i.e. $1/\nu(\sigma) = 0$, and the line $\sigma \mapsto a(\sigma)$ is neutral, i.e. it is not elongated, then $\alpha(\sigma) = 1$, and we arrive at the classical equations of the statics of a rectilinear beam.

The theory of an inextensible perfectly flexible string. Since the topic of discussion is the plane equilibrium a perfectly flexible string, each section is considered as a point. We have

$$\Theta^c = \mathrm{col}[0, 0, 0, N, 0, 0], \quad e^c = \mathrm{col}[0, 0, 0, 0, 0, 1/\nu(\sigma)]$$

Equations (3.2) in this case take the following classical form

$$F + \frac{dN}{d\sigma} = 0, \quad P + \frac{1}{\nu}N = 0$$

Suppose the external action is its own weight. We put

$$\mathbf{t} = \cos\beta\mathbf{i}_1 + \sin\beta\mathbf{j}_1$$

Then

$$-p\sin\beta + \frac{dN}{d\sigma} = 0, \quad -p\cos\beta + \frac{1}{\nu}N = 0$$

where $p = p(\sigma)$ is the mass density. Integration of these equations (for constant p) gives the classical catenary.

In a similar way one can investigate a set of other models. For example, in Timoshenko's model one degree of freedom is added to the displacements, associated with the Frenet trihedron, already used in Section 2, denoted as $D_f(\sigma, t)$. Then, for example, the total displacement is specified as

$$D(\sigma, t) = D_f(\sigma, t) \circ \exp(\psi\zeta_1)$$

We can also use other representations, but then one must bear in mind the non-commutativity in the non-linear approximation.

Automatic derivation of the scalar equations of motion of the system considered. By virtue of the fact that there are several possible representations of the displacements, we will dwell on the automatic derivation of the scalar equations of motion of the system considered when there is a certain freedom in choosing the representation of the mapping $(\sigma, t) \mapsto D(\sigma, t)$ and freedom in choosing the initial configuration $r(\sigma)$. We will present the main elements of this program.

1. In general, the position of the system is defined by two parameters $w = (\sigma, t)$, where σ is the curvilinear abscissa of the section and t is the time.

2. We introduce three bases in the space $\mathcal{D}$, recalling that with each system of coordinates $\mathcal{R} = (A; \mathbf{i}, \mathbf{j}, \mathbf{k})$ in the affine space $\mathcal{E}$ we associate a basis $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k}, \xi, \eta, \zeta)$ of space $\mathcal{D}$ according to the rule (3.1). The three bases $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ are associated with the three distributions of the affine systems of coordinates: the fixed system of coordinates $\mathcal{R}_0 = (A_0; \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$, the system of coordinates

$$\mathcal{R}_1 = \mathcal{R}_1(\sigma) = (A_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1) = (D_r(\sigma)(A_0); \mathbf{D}_r(\sigma)(\mathbf{i}_0), \mathbf{D}_r(\sigma)(\mathbf{j}_0), \mathbf{D}_r(\sigma)(\mathbf{k}_0))$$

which defines the initial configuration $r(\sigma)$, and the system of coordinates

$$\mathcal{R}_2 = \mathcal{R}_2(\sigma, t) = (D(\sigma, t)(A_1); \mathbf{D}(\sigma, t)(\mathbf{i}_1), \mathbf{D}(\sigma, t)(\mathbf{j}_1), \mathbf{D}(\sigma, t)(\mathbf{k}_1))$$

which defines the actual configuration at the instant of time t.

3. The basis chosen to write the scalar equations will be the basis $\mathcal{B}$, which depends on σ, but this argument will be omitted. Moreover, in order to represent the element $D(\sigma, t)$, we will denote by ρ_1^i ($i = 1, 2, 3$) the three vectors chosen from the family (ξ_1, η_1, ζ_1) and we will write

$$D(\sigma, t) = \exp(u(\sigma, t)) \circ \exp(\psi_1\rho_1^1) \circ (\psi_2\rho_1^2) \circ \exp(\psi_3\rho_1^3)$$

where $u(\sigma, t) = x\mathbf{i}_1 + y\mathbf{j}_1 + z\mathbf{k}_1$ is a constant vector field such that $\exp(u(\sigma, t))$ is the translation part of $D(\sigma, t)$, while $\exp(\psi_i\rho_1^i)$ describes rotation by an angle φ_i about the ρ_1^i axis ($i = 1, 2, 3$). The family

ρ_1^i (i = 1, 2, 3) is chosen arbitrarily, but must allow rotation to be described. The scalar unknowns of the problem ($x, y, z, \psi_1, \psi_2, \psi_3$) are functions of t and σ.

4. The elements of group $\mathbb{D}$ are represented solely by its adjoint representation. Since this representation is a group morphism, it is sufficient to express Ad(exp$X$) only for $X \in \mathcal{B}_1$ and to specify its matrix in $\mathcal{B}_1$.

If u = col[a, b, c, 0, 0, 0] is a constant vector field in $\mathcal{B}_1$, we have

$$\mathrm{Ad}(\exp u) = \begin{Vmatrix} \mathbf{I} & \mathbf{\Lambda}_1 \\ \mathbf{O} & \mathbf{I} \end{Vmatrix}, \quad \mathbf{\Lambda}_1 = \begin{Vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{Vmatrix}$$

where **I** is the identity matrix and **O** is the zero 3 × 3 matrix.

If $u = \xi_1$ = col[0, 0, 0, 1, 0, 0], we have

$$\mathrm{Ad}(\exp \alpha u) = \begin{Vmatrix} \mathbf{\Lambda}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}_2 \end{Vmatrix}, \quad \mathbf{\Lambda}_2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{Vmatrix}$$

If $u = \eta_1$ = col[0, 0, 0, 0, 1, 0], we have

$$\mathrm{Ad}(\exp \alpha u) = \begin{Vmatrix} \mathbf{\Lambda}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}_3 \end{Vmatrix}, \quad \mathbf{\Lambda}_3 = \begin{Vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{Vmatrix}$$

If $u = \zeta_1$ = col[0, 0, 0, 0, 0, 1], we have

$$\mathrm{Ad}(\exp \alpha u) = \begin{Vmatrix} \mathbf{\Lambda}_4 & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}_4 \end{Vmatrix}, \quad \mathbf{\Lambda}_4 = \begin{Vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

5. For the Lie bracket the matrix of the brackets has the form

$$\begin{Vmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_2 & \mathbf{O} \end{Vmatrix}, \quad \mathbf{L}_1 = \begin{Vmatrix} 0 & \xi_1 & -\eta_1 \\ -\zeta_1 & 0 & \xi_1 \\ \eta_1 & -\xi_1 & 0 \end{Vmatrix}, \quad \mathbf{L}_2 = \begin{Vmatrix} 0 & \mathbf{k}_1 & -\mathbf{j}_1 \\ -\mathbf{k}_1 & 0 & \mathbf{i}_1 \\ \mathbf{j}_1 & -\mathbf{i}_1 & 0 \end{Vmatrix}$$

The remaining components are found from the bilinearity.

6. As regards the differentiations we note that if $\upsilon(t, \sigma) = \upsilon_i(t, \sigma)$, ($i$ = 1, ... , 6) is a vector form $\mathfrak{D}$, specified in the basis $\mathcal{B}_1$, we have

$$\frac{\partial \upsilon(t,\sigma)}{\partial t} = \frac{\partial \upsilon_i(t,\sigma)}{\partial t}, \quad \frac{\partial \upsilon(t,\sigma)}{\partial \sigma} = \frac{\partial \upsilon_i(t,\sigma)}{\partial \sigma} + [\omega, \upsilon], \quad i = 1, \dots, 6$$

where the vector ω belongs to the data of the problem considered and is a function in $D_r(\sigma)$, defining the initial configuration of the system.

7. Suppose $w = t$ or $w = \sigma$. We put successively (i = 1, 2, 3)

$$\mathbf{A}_0 = \mathrm{Ad}(\exp u), \quad \mathbf{A}_i = \mathrm{Ad}(\exp(u)) \circ \dots \circ \mathrm{Ad}(\exp(\psi_i \rho_1^i))$$

$$\omega_0 = \frac{\partial u}{\partial w}, \quad \omega_i = \frac{\partial \psi_i}{\partial w}\rho_1^i + \mathrm{Ad}(\exp(\psi_i \rho_1^i) - 1)\omega_w$$

Suppose

$$\omega^c = \mathrm{Ad}(D)^{-1}\left(\sum_{i=0}^{3} \mathbf{A}_i\omega_i\right)$$

Then, if $w = t$, we have $\omega_t = 0$ and $\omega^c = V^c$, while if $w = \sigma$, we have $\omega_\sigma = \omega$ and $\omega^c = e^c$.

8. The matrix $\mathbf{H}_r$ of the inertia operator H_r in the basis $\mathcal{B}_1$ depends on the data of the problems. For example, if the origin of the system of coordinates $\mathcal{R}_1$ coincides with the centre of mass of the corresponding section and if $\mathbf{J}$ is the matrix of the central tensor of inertia in the basis $(\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1)$, we have

$$\mathbf{H}_r = \begin{Vmatrix} \mathbf{O} & \mathbf{J} \\ \mathbf{I} & \mathbf{O} \end{Vmatrix}$$

9. The coordinates θ_i of the vector Θ^c in the basis $\mathcal{B}_1$ specify dynamic variables of the system.

As an example we will give an expression for one of the six scalar equations obtained (we have chosen the projection onto the η_1 axis, and a prime denotes a partial derivative with respect to σ while a dot denotes a derivative with respect to t).

$$\begin{aligned}
\mathcal{T}_4^c &= (\theta_3 c_2 s_3 - \theta_2 c_2 c_3)z' + \\
&+ (\theta_3(c_1 c_3 + s_1 s_2) + \theta_2(c_1 s_3 - c_3 s_1 s_2))y' + \\
&+ (\theta_3(c_1 s_2 - c_3 s_1)) - \theta_2(c_1 c_3 s_2 + s_1 s_3)x' + \\
&+ \rho I_{11}(\ddot\psi_3 - \dot\psi_2^2 c_2 - \ddot\psi_1 s_2) + \rho(I_{22} - I_{33})[\dot\psi_2^2 c_2(s_3^2 - c_3^2) + (\dot\psi_2^2 - \dot\psi_1^2 c_2^2 c_3 s_3)] - \\
&- \theta_2(c_1 s_2 + s_1 s_3) + \theta_3(c_1 s_2 - c_3 s_1) + \theta_5(\psi_2' - c_2\psi_1') + \theta_6(\psi_2' + c_2\psi_1') + \theta_4'
\end{aligned}$$

where I_{ij} are the components of the inertia tensor of the section and, for brevity, we have introduced the following notation

$$c_i = \cos\psi_i, \quad s_i = \sin\psi_i; \quad i = 1, 2, 3$$

The complex form of this equation indicates that it would be difficult to obtain it without using the coordinateless approach described above and without indicating a program for deriving these equations automatically.

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